

# An exact solution of thermal stability analysis of bimorph functionally graded annular plates

## Solución exacta de análisis de estabilidad térmica de placas anulares graduadas funcionalmente dimorfas

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**Manuscript Code:** 665

**Date of Acceptance/Reception:** 10.04.2017/09.08.2015

**DOI:** 10.7764/RDLC.16.1.66

### Abstract

The present study aims to provide an exact solution for thermal buckling of bimorph functionally graded circular plates under uniform thermal loading with regard to von Kármán's classic theory and non-linear assumptions in clamped-clamped, simple-simple, clamped-simple and simple-clamped support conditions. Material properties will change in association to the middle surface of symmetric plate and according to the power law in direction of thickness. So that, the middle surface of the annular plate was pure metal and the sides of plate were pure ceramic. Using non-linear equations energy, the equilibrium was determined and the stability equations were employed by the method of equilibrium in vicinity in order to specify the critical temperature of buckling. Finally, a closed solution was obtained. We also measured the impact of varying factors like the rate of thickness to plate radius changes, volume fraction percentile changes of materials, and the ratio of the inner radius to the outer radius over the critical buckling temperature. Then, the results were compared together and with former studies. The findings indicated that for bimorph functionally graded materials, thermal moment does not occur, while buckling critical temperature in bimorph functionally graded materials of annular plates increases as the ratio of thickness to radius increases. Moreover, by increasing the ratio of inner to outer radius in clamped-clamped and clamped-simple support conditions we have an increase in thermal buckling free parameter.

**Key words:** Thermal buckling, annular plates, functionally graded materials, classical theory

### Resumen

El presente estudio tiene por objeto proporcionar una solución exacta para el pandeo térmico de placas bimorfas funcionalmente graduadas circulares bajo carga térmica uniforme con respecto a la teoría clásica de von Kármán y suposiciones no lineales en Condiciones de soporte: empotrado-empotrado, articulado-articulado, empotrado-articulado y articulado-empotrado. Las propiedades de los materiales cambiarán en asociación con la superficie media de la placa simétrica y de acuerdo con la ley de potencia en la dirección del espesor. De manera que la superficie media de la placa anular era de metal puro y los lados de la placa eran de cerámica pura. Utilizando ecuaciones no lineales de energía, se determinó el equilibrio y se emplearon las ecuaciones de estabilidad mediante el método de equilibrio en la vecindad para especificar la temperatura crítica de pandeo. Finalmente, se obtuvo una solución cerrada. También medimos el impacto de factores variables como la tasa de espesor de los cambios en el radio de la placa, los cambios en el percentil de la fracción volumétrica de los materiales y la relación entre el radio interior y el radio exterior sobre la temperatura crítica de pandeo. A continuación, los resultados se compararon entre sí y con estudios anteriores. Los resultados indicaron que para los materiales bimórficos funcionalmente clasificados, no ocurre el momento térmico, mientras que la temperatura crítica de pandeo en los materiales bimórficos funcionalmente clasificados de las placas anulares aumenta a medida que aumenta la relación entre el grosor y el radio. Además, al aumentar la relación entre el radio interior y el exterior en condiciones de sujeción sujetas con abrazadera y sujeción simple, tenemos un aumento en el parámetro libre de pandeo térmico.

**Palabras clave:** Pandeo térmico, placas anulares, materiales con clasificación funcional, teoría clásica.

## Introduction

Considering this fact that plates with various geometric shapes have been extensively applied in different industries, investigating their behavior is of great importance. Based on type of the application, plates may undergo thermal loading, which unauthorized imposing of the loads may lead to instability and breakdown of plates. Therefore, studying plates behavior has known to be one of central issues in engineering. The rate of minimum buckling load is central in engineering designs and calculations. Recently, functional materials have found special place in advanced industries including aerospace and nuclear industries. Thus, due to significance of the subject, a large body of research has been conducted. Generally speaking, functional materials are typically made of ceramic mixture with metal and/or a combination of different metals. These materials are heterogeneous in terms of microstructure and such structural properties like type of distribution and phases' size roundly change and this gradual change results in gradual transformations in mechanical and thermal properties of the materials (Koizumi, 1993).

Stability analysis and examining plates buckling behavior have drawn the scholars' attention as one of important problems in the analysis of structures. The first stability problem solution was presented by Brayan (1891). In his study, an annular plate buckling with clamped support was tested under uniform radial load. Yamaki (1958) worked on buckling of annular plates with loading on the internal and external edge. He showed that buckling does not necessarily occur in the first conditions. Timoshenko & Gere (1961) investigated the stability problem of different engineering structures like bars, frames, curved beams, plates and shells. Then, Brush and Almroth (1975) introduced a compressive analysis of buckling problem of bars, plates, shells and different methods for formulating non-linear equilibrium and stability equations.

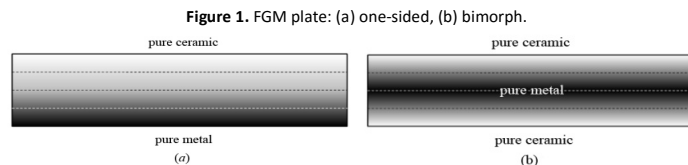
Reddy & Khedir (1989) analyzed buckling and free vibrations of rectangular multilayer plates using classical theory, first-order and third-order shear theory in boundary condition. Additionally, numerical solution based on finite element method was also performed. The results indicated that the classical theory can estimate natural frequencies and critical buckling load greater than actual value. Moreover, by increase of ratio of thickness to side length, the disparity increases, as well. Ozakca, Taysi & Kolcu (2003) benefiting from the finite element method and shear effects in form of first-order shear theory, studied buckling analysis and optimization of annular and circular plates thickness.

Kadkhodayan, Zhnag & Sowerby (1997) examined buckling analysis of annular and circular plates as well as rectangular plates based on the classical theory adopting dynamic relaxation method. Xu, Wang & Chen (2005) utilized 3D theory of elasticity for analyzing mechanical buckling of circular and annular plates. Then, they computed the results for different Poisson coefficients and compared them together.

Najafizadeh & Eslami (2002a,b) worked on buckling analysis of circular plates made of functionally graded materials under different thermal loading for clamped boundary conditions. Additionally, for mechanical loading, they measured simple and clamped boundary conditions under uniform radial compression. These studies employed the classical theory and provided an analytical solution. Naei, Masoumi & Shamekhi (2007) investigated buckling analysis of circular functionally graded material plate having variable thickness under uniform radial compression by finite element method and using the classical theory. The boundary conditions for circular plates were simple and clamped.

Sepahi, Forouzan & Malekzadeh (2011) worked on thermal buckling and post-buckling analysis of functionally graded annular plates with distribution of properties in radial direction in different supportive circumstances. The numerical solution was presented on the basis of differential quarature method. The results were presented in first-order shear theory and temperature-dependend material properties and in a radial direction. Golmakani & Emami (2013) studied nonlinear bending and buckling analysis of functionally graded annular plates with properties distribution in radial direction under mechanical loading in clamped and simple support circumstances. The numerical solution was provided on the basis of dynamic relaxation method. The results were presented for first-order shear theory.

Functionally graded annular plates are classified into two groups of one way and two-way groups in terms of the circumstances with high temperature are imposed on one side or both sides of the plate. Figure 1 shows the classification. In one-way state, one side is pure ceramic and the other side is pure metal. However, in two-way state, both sides usually are of pure ceramic and the middle plate is of pure metal. In annular plates with functionally graded materials with simple support under uniform thermal loading, due to asymmetrical materials in relation to the middle surface, thermal moment creates in the plate, which makes equations heterogeneous and lack of access to Eigenvalues as buckling critical temperature. In a physical viewpoint, presence of this thermal moment causes the plate to bind with the minimum possible load and so the lack of buckling behavior. In this regard, describing thermal buckling load does not make a sense (Li et al., 2007). But, in bimorph functionally graded materials due to presence of symmetrical materials versus the middle surface, thermal moment does not appear and we can examine buckling behavior in simple support circumstances.



In previously performed studies, functionally graded materials were analyzed, yet bimorph functionally graded materials has remained less recognized. Here, the authors found out that resistance towards bimorph functionally graded buckling is greater than functionally graded material. So, we could examine thermal buckling in bimorph functionally graded materials under uniform thermal loading in simple support circumstances.

Eskandari-Jam et al. (2013) provided an exact solution of mechanical buckling of bimorph functionally graded circular plates with regard to the classical theory and presented a closed solution. Then, they compared the results with functionally graded materials. In this research, the author realized that in the bimorph functionally graded materials resistance against clamped support buckling is three times more than simple support circumstances.

Jabbarzadeh, Eskandari-Jam & Khosravi (2012) analyzed thermal buckling behavior of circular plates of variable thickness from bimorph functionally graded materials under uniform thermal loading based on the first order shear theory in simple and clamped support circumstances. At the end, they calculated an optimal value for thermal buckling parameter in clamped support circumstances.

In another research with a main focus on vibration analysis of bridges, a dynamic monitoring model based on non-linear modelling was introduced for identifying the damages that may occur in the in civil engineering structures. Applying such modelling on the structures yields the structural conditions of bridges. Moreover, it was demonstrated that the nonlinearity is beneficial for investigation of the structural condition which is so essential during decision-making process on structural retrofitting (Consuegra & Santos, 2015).

Another study was also accomplished concerning the curtain walls which were investigated in terms of thermal performance. During the afore-mentioned research, the thermal transmittance and risk of condensation were studied by making a comparison between conventional system and analytical and numerical thermal analysis (Cordero, Garcia-Santos & Overend, 2015).

The available studies have been performed on thermal and mechanical buckling of functionally graded annular plates or changes of properties in radial direction as well as numerical solution. But, no independent study has investigated so far thermal buckling analysis of bimorph annular plates with properties variations in thickness direction and under different support circumstances in form of analytical solution. In the present study, the classical theory and von Kármán's assumptions for thermal buckling analysis of bimorph functionally graded materials of annular plates under uniform thermal loading. To obtain non-linear equations of equilibrium, we used the minimum potential energy principle and adopting the turbulence theory and non-linear equations of equilibrium, and the equations became linear. For determination of buckling critical temperature, stability equations were computed by equilibrium method in the vicinity obtained from the linear equations of equilibrium. Then, simple-simple, clamped-clamped, simple-clamped and clamped-simple support conditions were analytically solved and a close solution was provided.

The impact of different factors including the changing rate of thickness to plate radius, percent variation of volume fraction of materials, and change in internal radius ratio to external radius in different support circumstances under critical buckling temperature were discussed in details. Then, the results were provided in tables and diagrams. In order to verify the results, we compared them with previously reported results.

## Formulation

### The problem geometry

The geometry used for measuring thermal buckling of annular plates with internal radius  $b$ , external radius  $a$ , thickness  $h$ , are under uniformly increasing thermal loading.

### Properties of functional graded materials

The study plate is of bimorph functionally graded materials in which volume fraction of ceramic and corresponding the materials properties are symmetrical in relation to middle surface of the plate. It changes in direction to thickness constantly in a way that the middle surface of the plate is pure metal and by approximating to outer surfaces, the

ceramic proportion increases. Finally, upper and lower surfaces .i.e.  $z = \frac{h}{2}$  and  $z = -\frac{h}{2}$  are totally made of ceramic. Since functionally graded materials are a combination of metal and ceramic, the properties of these materials like modulus of elasticity  $E$ , coefficient of thermal expansion  $\alpha$  according to the classical linear combinations, are stated as follows:

$$\begin{aligned}
E_f &= E_m V_m + E_c V_c \\
\alpha_f &= \alpha_m V_m + \alpha_c V_c
\end{aligned}
\tag{1}$$

Where, c= properties of ceramic, m=properties of metal and  $V_c$  and  $V_m$  = volume fraction. The properties of bimorph functionally graded materials with exponential equations are formulated as below:

$$\begin{aligned}
P^1(z) &= P_m + (P_c - P_m) \left(-\frac{2z}{h}\right)^n, \quad -\frac{h}{2} \leq z \leq 0 \\
P^2(z) &= P_m + (P_c - P_m) \left(\frac{2z}{h}\right)^n, \quad 0 \leq z \leq \frac{h}{2}
\end{aligned}
\tag{2}$$

where, P= properties of materials, and as a module of elasticity or coefficient of thermal expansion. The variation of v is insignificant and we consider it constant. Also, n is volume fraction of functionally graded materials showing integration of ceramic and metal volume fraction in direction to thickness. It could be greater or equal to zero. The zero and infinite values are pure ceramic and pure metal, respectively (Jabbarzadeh et al., 2012).

### Equations of stability and equilibrium

The displacement field defines in terms of the classical theory of plates in polar coordinates with axial symmetry as follows (Eskandari-Jam et al., 2013):

$$\begin{aligned}
U(r, z) &= u(r) - z \frac{dw}{dr} \\
V &= 0, \quad W(r, z) = w(r)
\end{aligned}
\tag{3}$$

where, U, V and W are displacement in direction to  $z, \theta, r$  and  $w(r), u(r)$  are the displacement of the middle plates in direction to z, r. The problem of buckling is a sub-series of non-linear problems of geometry. But, in non-linear problems displacement is in area which is impossible to overlook non-linear statements in strain-displacement relations. The strain-displacement relations are defined in terms of von Karmen assumptions and the Equation (3) as below:

$$\varepsilon_r = \frac{dU}{dr} + \frac{1}{2} \left( \frac{dW}{dr} \right)^2, \quad \varepsilon_\theta = \frac{U}{r}, \quad \gamma_{r\theta} = 2\varepsilon_{r\theta} = 0
\tag{4}$$

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{r0} \\ \varepsilon_{\theta0} \\ \gamma_{r\theta0} \end{Bmatrix} + z \begin{Bmatrix} k_r \\ k_\theta \\ k_{r\theta} \end{Bmatrix}
\tag{5}$$

The strains of middle plate describes as below:

$$\varepsilon_{r0} = \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2, \quad \varepsilon_{\theta0} = \frac{u}{r}, \quad \gamma_{r\theta0} = 0
\tag{6}$$

The curvatures are described as below:

$$k_r = -\left(\frac{d^2w}{dr^2}\right), \quad k_\theta = -\frac{1}{r}\left(\frac{dw}{dr}\right), \quad k_{r\theta} = 0 \quad (7)$$

Since functionally graded materials are isotropic and overlooking stress in thickness direction and constant Poisson's coefficient, stress-strain relations in terms of the Hooke's law as below:

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) - \frac{E\alpha\Delta T}{1-\nu} \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) - \frac{E\alpha\Delta T}{1-\nu} \\ \tau_{r\theta} &= \frac{E}{2(1+\nu)}\gamma_{r\theta} = 0 \end{aligned} \quad (8)$$

where, E= modulus of elasticity,  $\alpha$  = coefficient of thermal expansion for functional materials,  $\sigma_\theta, \sigma_r$  =normal stress,  $\tau_{r\theta}$  = shear stress at any point of the plate thickness in each point of thickness with z distance from middle plate.

With the integration of the components of stress on the plate thickness, the resultant forces and moments in terms of stress components are as follows:

$$\begin{aligned} (N_r, N_\theta) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_r, \sigma_\theta) dz \\ (M_r, M_\theta) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_r, \sigma_\theta) z dz \end{aligned} \quad (9)$$

Replacing Equation (8) with Equation (9), the relationship between forces and moments in terms of strain components are as below:

$$\begin{aligned} \begin{pmatrix} N_r \\ N_\theta \end{pmatrix} &= A \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_{r0} \\ \varepsilon_{\theta0} \end{pmatrix} - \begin{pmatrix} N_r^T \\ N_\theta^T \end{pmatrix} \\ \begin{pmatrix} M_r \\ M_\theta \end{pmatrix} &= B \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{pmatrix} k_r \\ k_\theta \end{pmatrix} - \begin{pmatrix} M_r^T \\ M_\theta^T \end{pmatrix} \end{aligned} \quad (10)$$

Also, by replacing Equation (6) and (7) with Equation (10), the relationship between forces and moments in terms of displacement components are as below:

$$\begin{aligned}
N_r &= A \left( \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 + \nu \frac{u}{r} \right) - N_r^T \\
N_\theta &= A \left( \frac{u}{r} + \nu \left( \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2 \right) \right) - N_\theta^T \\
M_r &= B \left( -\frac{d^2w}{dr^2} - \frac{\nu}{r} \frac{dw}{dr} \right) - M_r^T \\
M_\theta &= B \left( -\frac{1}{r} \frac{dw}{dr} - \nu \frac{d^2w}{dr^2} \right) - M_\theta^T
\end{aligned} \tag{11}$$

Coefficients A, B, through integration of properties in direction to thickness as below:

$$(A, B) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z^2) \frac{E(z)}{1-\nu^2} dz \tag{12}$$

The resultant thermal forces and moments are created because of thermal loading in form of increased temperature  $\Delta T$  on the plate. These thermal forces and moments are computed as follows:

$$\begin{aligned}
(N_r^T, M_r^T) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z)}{1-\nu} \alpha(z) \Delta T (1, z) dz \\
(N_\theta^T, M_\theta^T) &= (N_r^T, M_r^T)
\end{aligned} \tag{13}$$

The equilibrium equations for annular plates could be computed via minimizing the potential energy or formulating the equilibrium equations for only one element. Due to axial symmetry, there is no change in environmental direction and only derivatives appear with respect to radial direction of differential equations set.

$$\begin{aligned}
\frac{dN_r}{dr} + \frac{N_r - N_\theta}{r} &= 0 \\
-\frac{1}{r} \frac{d}{dr} (rN_r \frac{dw}{dr}) - \frac{d^2M_r}{dr^2} - \frac{2}{r} \frac{dM_r}{dr} + \frac{1}{r} \frac{dM_\theta}{dr} &= 0
\end{aligned} \tag{14}$$

To calculate the sustainability equations from non-linear equations of equilibrium, the scholars use equilibrium in the vicinity (Brush & Almroth, 1975). This method is used for evaluating sustainability and analysis of structures buckling behavior and sustainability as well as critical load of buckling based on primary and secondary sustainability paths and ramifications points are utilized. Using this technique, we are able to calculate the ramification point by solving linear differential equations. That is, for a balanced state on initial balance path, we consider the probability of existence of adjacent balanced shape under the same load. Such a balanced shape in adjacency of primary equilibrium indicates presence of a ramification point on the balanced path. Points created due to crossed balanced paths are called ramification points. Here, the equilibrium equations have two solutions each of which corresponds to a branch.

The required equations for this method derive from non-linear equilibrium equations using the turbulence method. Where, displacement fields  $(u, w)$  are replaced with the field  $(u_0 + u_1, w_0 + w_1)$ , which  $(u_0, w_0)$  shows the initial equilibrium state prior to buckling. It also indicates a balanced state on the primary path and  $(u_1, w_1)$  is small

displacements in the displacement field. But, we should mention here that in pre-buckling state, the annular plate does not bend or displace and so  $w_0 = 0$ .

With replacing the new fields in Equation (14), all statements do not include small displacement which are removed from the resulted equations.

Moreover, if increased displacement is small enough, only the first order statements of displacement  $(u_1, w_1)$  remain in equations and higher order statements are removed. Therefore, the resulted sustainability equations obtained by linear and homogenous equations with respect to small displacement field are hypothetical. This instruction is mostly known for determination of stability as a standard for balanced stability in the vicinity. Therefore, the stability equations are stated as follows:

$$\begin{aligned} \frac{dN_{r1}}{dr} + \frac{N_{r1} - N_{\theta1}}{r} &= 0 \\ -\frac{1}{r} \frac{d}{dr} (rN_{r0} \frac{dw_1}{dr}) - \frac{d^2 M_{r1}}{dr^2} - \frac{2}{r} \frac{dM_{r1}}{dr} + \frac{1}{r} \frac{dM_{\theta1}}{dr} &= 0 \end{aligned} \quad (15)$$

where:

$$\begin{aligned} N_{r1} &= A \left( \frac{du_1}{dr} + \nu \frac{u_1}{r} \right) \\ N_{\theta1} &= A \left( \frac{u_1}{r} + \nu \frac{du_1}{dr} \right) \\ M_{r1} &= B \left( -\frac{d^2 w_1}{dr^2} - \frac{\nu}{r} \frac{dw_1}{dr} \right) \\ M_{\theta1} &= B \left( -\frac{1}{r} \frac{dw_1}{dr} - \nu \frac{d^2 w_1}{dr^2} \right) \end{aligned} \quad (16)$$

$$\begin{aligned} N_{r0} &= A \left( \frac{du_0}{dr} + \nu \frac{u_0}{r} \right) - N_{r0}^T \\ N_{\theta0} &= A \left( \frac{u_0}{r} + \nu \frac{du_0}{dr} \right) - N_{\theta0}^T \\ M_{r0} &= B \left( -\frac{d^2 w_0}{dr^2} - \frac{\nu}{r} \frac{dw_0}{dr} \right) - M_{r0}^T = 0 \\ M_{\theta0} &= B \left( -\frac{1}{r} \frac{dw_0}{dr} - \nu \frac{d^2 w_0}{dr^2} \right) - M_{\theta0}^T = 0 \end{aligned} \quad (17)$$

By replacing Equation (16) with Equation (15), the stability equations with respect to displacement are as follows:

$$\begin{aligned} A \left( \frac{d^2 u_1}{dr^2} + \frac{1}{r} \frac{du_1}{dr} - \frac{u_1}{r^2} \right) &= 0 \\ B \nabla^4 w_1 - \frac{1}{r} \frac{d}{dr} (rN_{r0} \frac{dw_1}{dr}) &= 0 \end{aligned} \quad (18)$$

As we can see, the resulted stability equations are independent. Therefore, to calculate the critical temperature of buckling, we use the second equation. In the stability equations,  $N_{r0}$  radial force is pre-buckling is determined via non-linear equilibrium equations .so, by replacing equation (17) with the equilibrium equations (14), we have:

$$\frac{d^2 u_0}{dr^2} + \frac{1}{r} \frac{du_0}{dr} - \frac{u_0}{r^2} = 0 \quad (19)$$

Equation (19) is the plate membrane equation. This is a usual linear differential equation by computing it we can determine displacement and interpolate forces of pre-buckling. The independent variables of the equation is  $u_0$ , in-place forces are computed. The solution for Equation (19) is as below:

$$u_0 = c_1 r + \frac{c_2}{r} \quad (20)$$

Thermal loading is imposed through uniform increase of plate temperature from a reference temperature. In this state, the edge has no freedom to displace in radial direction and avoidance from increased plate length causes a compression force. Moreover, due to asymmetry, disbarment in center must be limited. As a result, the boundary condition for Equation (20) is as follows:

$$\begin{aligned} \text{in the inner edge } u_0(b) &= 0 \\ \text{in the outer edge } u_0(a) &= 0 \end{aligned} \quad (21)$$

Imposing boundary conditions in Equation (12), we have  $c_1 = c_2 = 0$  and so Equation (20) is obtained as below:

$$u_0 = 0 \quad (22)$$

Replacing Equation (22) with Equation (17), pre-buckling forces are computed as below:

$$N_{r0} = N_{\theta 0} = -N_{r0}^T \quad (23)$$

Replacing Equation (23) with the second satiability equation, we have:

$$B \nabla^4 w_1 = -\frac{1}{r} \frac{d}{dr} (r N_{r0}^T \frac{dw_1}{dr}) \quad (24)$$

Integrating the above equation once with respect to r, we obtain:

$$r^2 \alpha'' + r \alpha' + \alpha (\lambda^2 r^2 - 1) = 0 \quad (25)$$

where:



$$\alpha = -\frac{dw_1}{dr}, \quad \lambda^2 = \frac{N_{r0}^T}{B} \quad (26)$$

Now, we add one new variable as below:

$$s = \lambda r \quad (27)$$

Using that, Equation (25) is as follows:

$$s^2 \frac{d^2 \alpha}{ds^2} + s \frac{d \alpha}{ds} + (s^2 - 1) \alpha = 0 \quad (28)$$

The total solution to the equation is as below:

$$\alpha = c_5 J_1(s) + c_6 Y_1(s) \quad (29)$$

where,  $J_1$  and  $Y_1$  are the first order Bessel functions of the first and the second kind and  $c_5, c_6$  are constants of integration. Moreover, the clamped boundary conditions in the inner and outer edge are as below:

$$w_{1,r}(b) = 0, \quad w_{1,r}(a) = 0 \quad (30)$$

Considering the support conditions in both edges in clamped form and imposing support conditions (30) in equation (29) we have:

$$\begin{aligned} c_3 J_1(\lambda a) + c_4 Y_1(\lambda a) &= 0 \\ c_3 J_1(\lambda b) + c_4 Y_1(\lambda b) &= 0 \end{aligned} \quad (31)$$

As we see, we have a set of Eigenvalues. The condition for obtaining solution to this set is the Eigenvalues. In which Determinant coefficients equal to zero. Now, the solution for Set (31) states as below:

$$J_1(\lambda a) Y_1(\lambda b) - J_1(\lambda b) Y_1(\lambda a) = 0 \quad (32)$$

Solving the above equation per each inner and outer radius, the smallest root of this equation is  $\lambda$ . Replacing this value with Equation (26) we have:

$$N_{r0}^T = \lambda^2 B \quad (33)$$

Therefore, replacing Equation (33) with Equation (13), critical buckling temperature for annular plate in clamped-clamped support conditions is as follows:

$$\Delta T_{cr} = \lambda^2 \frac{B(1-\nu)}{I} \quad (34)$$

where:

$$I = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\alpha(z)dz \quad (35)$$

Moreover, boundary conditions of two edges in the simple form of inner and outer edges are as below:

$$w_{1,rr} + \frac{\nu}{b}w_{1,r} = 0, \quad w_{1,rr} + \frac{\nu}{a}w_{1,r} = 0 \quad (36)$$

Considering support conditions in both edges in simple form and imposing support conditions (36) in Equation (29) we have:

$$\begin{aligned} & [\lambda a J_0(\lambda a) - (1-\nu)J_1(\lambda a)][\lambda b Y_0(\lambda b) - (1-\nu)Y_1(\lambda b)] \\ & - [\lambda a Y_0(\lambda a) - (1-\nu)Y_1(\lambda a)][\lambda b J_0(\lambda b) - (1-\nu)J_1(\lambda b)] = 0 \end{aligned} \quad (37)$$

By solving the above equation and per each inner and outer radius, the smallest root is  $\lambda$  and by replacing this value in Equation (34), critical buckling temperature for annular plates in simple-simple support conditions is computed. Also, support conditions in simple inner edge and clamped outer edge will be as follows:

$$\left( w_{1,rr} + \frac{\nu}{b}w_{1,r} \right) = 0, \quad w_{1,r}(a) = 0 \quad (38)$$

By imposing support conditions (38) in Equation (29) we have:

$$\begin{aligned} & J_1(\lambda a)[\lambda b Y_0(\lambda b) - (1-\nu)Y_1(\lambda b)] \\ & - Y_1(\lambda a)[\lambda b J_0(\lambda b) - (1-\nu)J_1(\lambda b)] = 0 \end{aligned} \quad (39)$$

After solving the above equation and per each inner and outer radius, the smallest root is  $\lambda$  and by replacing this value in Equation (34), critical buckling temperature for annular plates in simple-clamped support conditions is computed. In addition, support conditions in clamped inner edge and simple outer edge will be as follows:

$$w_{1,r}(b) = 0, \quad \left( w_{1,rr} + \frac{\nu}{a}w_{1,r} \right) = 0 \quad (40)$$

By imposing support conditions (40) in Equation (29) we have:

$$\begin{aligned}
& Y_1(\lambda b) [\lambda a J_0(\lambda a) - (1-\nu) J_1(\lambda a)] \\
& - J_1(\lambda b) [\lambda a Y_0(\lambda a) - (1-\nu) Y_1(\lambda a)] = 0
\end{aligned}
\tag{41}$$

Subsequent to solving the above equation and per each inner and outer radius, the smallest root is  $\lambda$  and by replacing this value in Equation (34), critical buckling temperature for annular plates in clamped-simple support conditions is computed.

## Discussion and conclusion

In this section the results of exact solution of annular plates buckling of bimorph functionally graded materials in different support conditions under thermal loading will be presented. The functional material is a mixture of aluminum as a metal and alumina as ceramic were considered. Mechanical and thermal properties for aluminum and alumina are shown in Table 1 (Eskandari-Jam et al., 2013).

**Table 1.** Ingredients of bimorph functionally graded materials.

Coefficient of thermal expansion ( $1/^\circ\text{C}$ )	Yang module (Gpa)	Material
$23 \cdot 10^{-6}$	70	Aluminum
$7.4 \cdot 10^{-6}$	380	Alumina

To have a more precise evaluation of the results obtained from stability equations for annular plate, first by assuming the power law constant =0, bimorph functionally graded material changes into a homogenous material. Then, the parameter is calculated without thermal buckling  $\lambda_T$  and the results are compared with previously obtained results. In thermal loading, the parameter without buckling  $\lambda_T$  is calculated through Equation (42). Where,  $\nu$ ,  $\alpha_c$ , and  $\Delta T_{cr}$  are Poisson's ratio of the functional material, thermal expansion coefficient of ceramic and increase of the critical temperature.

$$\lambda_T = 12(1 + \nu) \Delta T_{cr} \alpha_c \left(\frac{a}{h}\right)^2
\tag{42}$$

Table (2) shows the parameter without thermal buckling of annular plate for different ratios h/a which is compared with reference values (Xu et al, 2005). So, the results indicate higher accuracy of findings obtained by the present method. After verifying the stability functions results for annular plates, the results of critical thermal buckling of annular plates for different values b/a are illustrated in Table (3).

Shown in table 3, by increase of the ratio of inner radius to outer radius in clamped-clamped and clamped-smile support conditions, we have an increase in thermal buckling free parameter. Moreover, in simple-simple support conditions we encounter reduced parameter of thermal buckling free. In simple-clamped support conditions, there is a decrease up to b/a=0.3 ratio, which above this value we have an increase in thermal buckling free parameter. Furthermore, in all ratios of inner radius to outer radius, for all lamped-clamped, simple-clamped, clamped-simple and simple-simple support conditions, they have the highest thermal buckling free parameter.

Figures 2, 3, 4 and 5 illustrate changes in buckling critical temperature  $\Delta T_{cr}$  of bimorph functionally graded materials in terms of the ratio of plate thickness to radius for different values of volume fraction index in clamped-clamped, simple-simple, clamped-simple and simple-clamped support conditions, Poisson's ratio 0.3 and ratio of inner to outer radius 0.3, respectively. Considering the results, we see that an increase in h/a of all support conditions, causes increased buckling critical temperature of bimorph functionally graded materials.

**Table 2.** Comparing thermal-buckling parameter with previous values.

BC	$h/a$	Reference	$b/a$		
			0.3	0.4	0.5
C-C <sup>a</sup>	0.001	Present	22.14432	29.06483	40.87245
		(Xu et al., 2005)	22.14432	29.06483	40.87245
	0.1	Present	22.14432	29.06483	40.87245
		(Xu et al., 2005)	20.55187	26.39141	35.80455
S-S. <sup>b</sup>	0.001	Present	2.60046	2.10869	1.75099
		(Xu et al., 2005)	2.60046	2.10869	1.75099
	0.1	Present	2.60046	2.10869	1.75099
		(Xu et al., 2005)	2.57673	2.09306	1.74019
S-C. <sup>c</sup>	0.001	Present	10.16050	10.88914	13.29086
		(Xu et al., 2005)	10.16050	10.88914	13.29086
	0.1	Present	10.16050	10.88914	13.29086
		(Xu et al., 2005)	9.80923	10.48686	12.69724
C-S. <sup>d</sup>	0.001	Present	5.78064	7.36288	10.13415
		(Xu et al., 2005)	5.78064	7.36288	10.13415
	0.1	Present	5.78064	7.36288	10.13415
		(Xu et al., 2005)	5.66490	7.17632	9.78467

<sup>a</sup>Both edges clamped.

<sup>b</sup>Both edges simply supported.

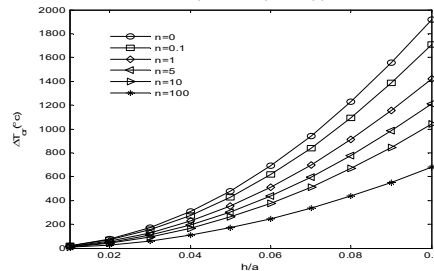
<sup>c</sup>Outer edges clamped, inner edges simply supported.

<sup>d</sup>Outer edges simply supported, inner edges clamped.

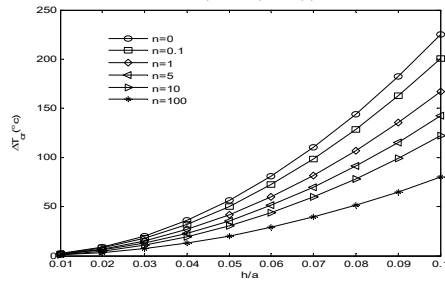
**Table 3.** Values of critical thermal-buckling parameter of homogenous annular plates.

BC	$b/a$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
C-C	15.53102	17.94156	22.14432	29.06483	40.87245	62.88633	110.71316	247.67054	987.79237
S-S	3.90689	3.24528	2.60046	2.10869	1.75099	1.48740	1.28842	1.13365	1.01036
S-C	13.14948	10.90686	10.16050	10.88914	13.29086	18.53927	30.50951	65.18093	252.03332
C-S	4.35948	4.85729	5.78064	7.36288	10.13415	15.39561	26.98165	60.52098	243.52147

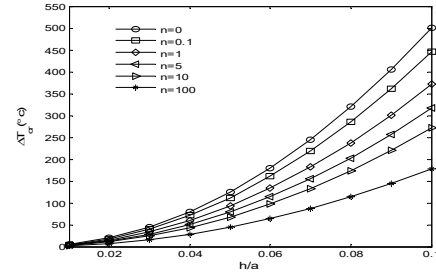
**Figure 2.** Changes of critical buckling temperature for different values of volume fractional index in clamped-clamped support conditions.



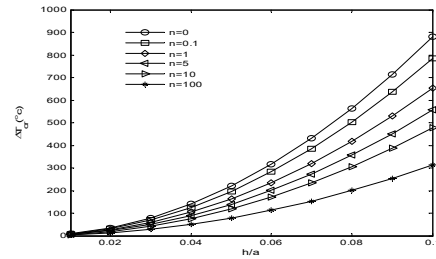
**Figure 3.** Changes of critical buckling temperature for different values of volume fractional index in simple-simple support conditions.



**Figure 4.** Changes of critical buckling temperature for different values of volume fractional index in clamped-simple support conditions.

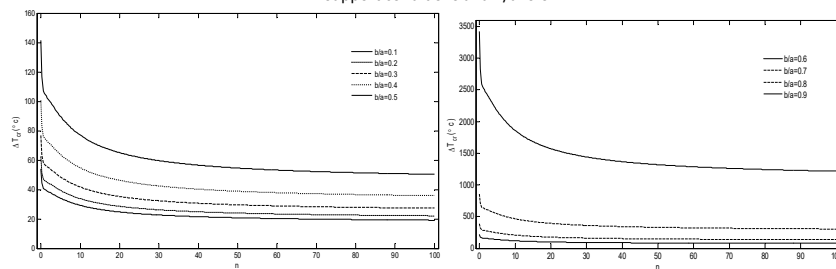


**Figure 5.** Changes of critical buckling temperature for different values of volume fractional index in simple-clamped support conditions.

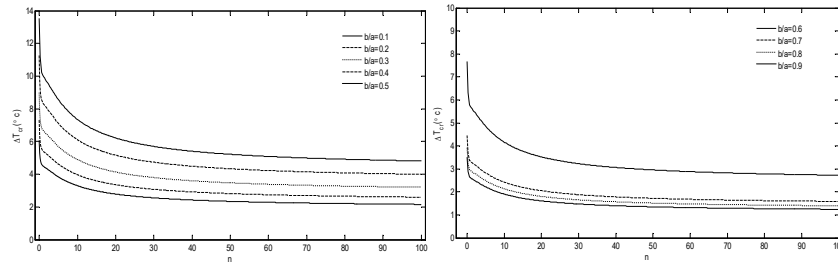


Figures 6, 7, 8 and 9 demonstrate changes in buckling critical temperature  $\Delta T_{cr}$  of bimorph functionally graded materials in terms of volume fraction index for different values of inner to outer radius ratio and the ratio of thickness to radius 0.2 in clamped-clamped, simple-simple, clamped-simple and simple-clamped support conditions, and Poisson's ratio 0.3, respectively. considering the results, we see that an increase in volume fraction index in entire ratios of inner to outer radius and support conditions, buckling critical temperature constantly decreases and in  $n=0$  (pure ceramic) it shows the highest value.

**Figure 6.** Changes of critical buckling temperature in terms of volume fractional index  $n$  for different values  $b/a$  in clamped-clamped support conditions and  $h/a=0.02$ .

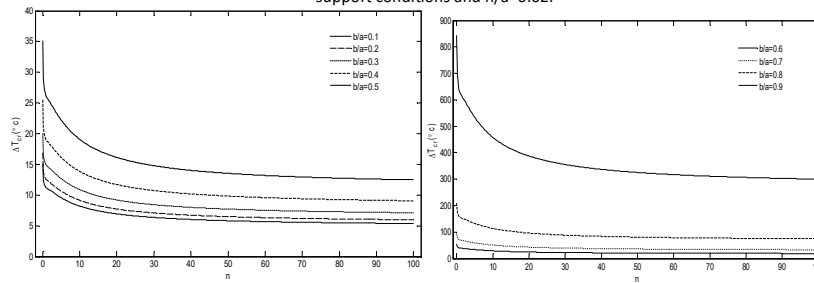


**Figure 7.** Changes of critical buckling temperature in terms of volume fractional index  $n$  for different values  $b/a$  in simple-simple support conditions and  $h/a=0.02$ .



The reason could be because increased volume fraction index leads to increased volume fraction index of metal on the plate and since ceramic stiffness is greater than metal stiffness, the overall plate stiffness falls and consequently, resistance of annular plate against buckling decreases. moreover, by increase of ratio of inner radius to outer radius in annular plates of bimorph functionally graded materials in clamped-clamped and clamped-simple conditions, we encounter with increased buckling critical temperature while in simple-simple support conditions, we have reduced critical temperature and in simple-clamped support conditions the temperature decreases up to  $b/a=0.3$  ratio and over this value, the buckling critical temperature increases.

**Figure 8.** Changes of critical buckling temperature in terms of volume fractional index  $n$  for different values  $b/a$  in clamped-simple support conditions and  $h/a=0.02$ .



**Figure 9.** Changes of critical buckling temperature in terms of volume fractional index  $n$  for different values  $b/a$  in simple-clamped support conditions and  $h/a=0.02$ .

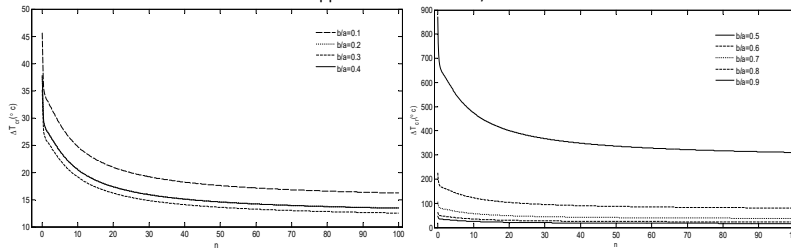
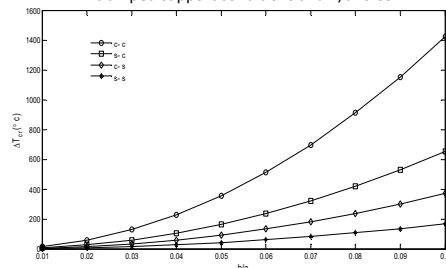


Figure 10 illustrates changes in buckling critical temperature of annular plates of bimorph functionally graded materials in terms of thickness to radius ratio for different support conditions and volume fraction index  $n=1$  and the ratio of inner to outer radius  $b/a=0.3$ . The results also show that resistance towards thermal buckling of annular plates of bimorph functionally graded materials in a constant volume fraction will have clamped-clamped, simple-clamped, clamped-simple and simple-simple support conditions with the highest buckling temperature.

**Figure 10.** Changes of critical buckling temperature in terms of volume fractional index  $n=1$  for different values  $b/a$  in clamped-clamped support conditions and  $h/a=0.03$ .



## Conclusions

In the present study, buckling exact solution of bimorph FGM under thermal loading in form of increasing temperature was measured in terms of the classical theory and Von-Carmen's non-linear displacement field in different support conditions. The stability equations were analytically solved and a closed solution was presented for determination of buckling critical temperature. The results are as follows:

In annular plates of functionally graded materials under uniform thermal loading, asymmetry of materials in relation to middle surface causes thermal moment before buckling in the plate and consequently leads to lack of access to buckling critical temperature. But, as the clamped support conditions bear bending moment, we can only assess the buckling behavior in clamped support conditions. However, for bimorph functionally graded materials due to asymmetry of materials in relation to middle surface, thermal moment does not occur and we can assess the buckling behavior in simple and clamped support conditions.

Buckling critical temperature in bimorph functionally graded materials of annular plates increases in clamped-clamped, simple-simple, clamped-simple and simple-clamped support conditions as the ratio of thickness to radius increases, as well. The buckling critical temperature in annular plates in clamped-clamped, simple-simple, simple-clamped and clamped-simple support conditions constantly decreases along with an increase in volume fraction index  $n$  and in  $n=0$  (pure ceramic) the temperature reaches its maximum. In annular plates, by constant ratios of inner to outer radius as well as increases ratio of thickness to radius, thermal buckling free parameter remains constant in all support conditions. By an increase in ratio of inner to outer radius in bimorph functionally graded materials in clamped-clamped and clamped-simple condition, we observe increase of buckling critical temperature. But, in simple-simple support conditions buckling critical temperature decreases and increases above this value. For entire inner and outer radius ratios in a constant volume fraction index, clamped-clamped, simple-clamped, clamped-simple and simple-simple support conditions show the highest buckling temperature, respectively.

By increasing the ratio of inner to outer radius in clamped-clamped and clamped-simple support conditions we have an increase in thermal buckling free parameter. Moreover, in simple-simple support conditions thermal buckling free parameter decreases and in simple-clamped support conditions it reduces up to  $b/a=0.3$  and it increases above this value.

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